On a generalization of certain results of Goriatchev, Chaplygin and Sretensky in the dynamics of rigid bodies

1996 J. Phys. A: Math. Gen. 29 8159

(http://iopscience.iop.org/0305-4470/29/24/036)

J. Phys. A: Math. Gen. **29** (1996) 8159–8161. Printed in the UK

On a generalization of certain results of Goriatchev, Chaplygin and Sretensky in the dynamics of rigid bodies

H M Yehia

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, and Department of Mathematics, Faculty of Science, King Abdulaziz University, PO Box 9028, Jeddah, Saudi Arabia†

Received 3 July 1996

Abstract. A new integrable case of the problem of motion of a gyrostat under the action of potential and gyroscopic forces is introduced. This case generalizes the classical Goriatchev–Chaplygin case of a heavy rigid body and its subsequent generalizations by Goriatchev and by Sretensky. A special version of the new case admits an interpretation as a problem of motion of a perforated body in a circulating fluid and equivalently as a case of motion of an electrically charged gyrostat acted upon by potential and Lorentz forces.

Analysis

Consider the Euler–Poisson equations:

$$\begin{aligned}
&4C\dot{p} - 3Cqr + q\mu_3 - r\mu_2 = \gamma_2\frac{\partial V}{\partial\gamma_3} - \gamma_3\frac{\partial V}{\partial\gamma_2}\\
&4C\dot{q} + 3Cpr + r\mu_1 - p\mu_3 = \gamma_3\frac{\partial V}{\partial\gamma_1} - \gamma_1\frac{\partial V}{\partial\gamma_3}\\
&C\dot{r} + p\mu_2 - q\mu_1 = \gamma_1\frac{\partial V}{\partial\gamma_2} - \gamma_2\frac{\partial V}{\partial\gamma_1}\\
&\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0, \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0, \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0
\end{aligned} \tag{1}$$

which describe the motion of a rigid body, with principal moments of inertia $A = B = 4C$, about a fixed point under the action of axisymmetric forces characterized by the potential $V = V(\gamma_1, \gamma_2, \gamma_3)$ and the vector $\boldsymbol{\mu}(\gamma_1, \gamma_2, \gamma_3)$.

It is straightforward to verify that for the choice

$$\begin{aligned}
&V = C\left[a\gamma_1 + b\gamma_2 + \nu k\gamma_3 + \frac{3\nu^2}{2\gamma_3^2} + \frac{\lambda}{\gamma_3^2}\right]\\
&\boldsymbol{\mu} = C(\nu\gamma_1, \nu\gamma_2, k + 7\nu\gamma_3)
\end{aligned} \tag{2}$$

where a, b, k, ν and λ are arbitrary parameters, the system (1) admits three general integrals:

$$\begin{aligned}
&I_1 = C\left[2(p^2 + q^2) + r^2/2\right] + V\\
&I_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2\\
&I_3 = 4(p\gamma_1 + q\gamma_2) + (r + k)\gamma_3 - \nu(4\gamma_1^2 + 4\gamma_2^2 + \gamma_3^2).
\end{aligned} \tag{3}$$

† Present address.

0305-4470/96/248159+03$19.50

It will be integrable whenever a fourth integral of motion, independent of the above ones, exists (see, e.g., [1]).

The main assertion of the present paper is that the system (1), (2) is integrable on the zero level of I_3. The fourth integral is

$$I_4 = (r - \nu\gamma_3 - k)\left[(p - \nu\gamma_1)^2 + (q - \nu\gamma_2)^2 + \frac{\lambda}{2\gamma_3^2}\right] - \gamma_3\left[ap + bq - \nu(a\gamma_1 + b\gamma_2)\right]. \quad (4)$$

In fact, by virtue of (1), (2), it can be shown that

$$\frac{\mathrm{d}I_4}{\mathrm{d}t} = \left[bp - aq + \nu(a\gamma_2 - b\gamma_1)\right] I_3.$$

The assertion follows immediately.

The present result unifies and generalizes some previously known integrable cases, which are all valid only on the zero level of the cyclic integral I_3:

(i) The Goriatchev–Chaplygin case of the motion of a heavy rigid body (see, e.g., Whittaker [4] and Leimanis [5], also Yehia [8]). It was originally found by Goriatchev [2] and generalized to its present form by Chaplygin [3]. In our notation this case is characterized by $V = Ca\gamma_1$, $\boldsymbol{\mu} = \mathbf{0}$.

(ii) Goriatchev's generalization of (i) by the inclusion of the singular term in the potential being $V = Ca\gamma_1 + \lambda/\gamma_3^2$, $\boldsymbol{\mu} = \mathbf{0}$ [6] (see also Yehia [8] for the derivation of the integral).

(iii) Sretensky's generalization of (i) to the problem of the motion of a heavy gyrostat [7]. That is, by the addition of gyrostatic moment along the axis of dynamical symmetry, so that $V = Ca\gamma_1$, $\boldsymbol{\mu} = (0, 0, Ck)$.

The singular term in the potential was introduced first by Goriatchev in two different problems [6, 10]. A similar term also appears in a class of integrable problems of motion of a body in an asymmetric field, as a result of ignoring a cyclic coordinate [11]. However, a direct physical interpretation of this term is not possible, since it means that a point of the axis of dynamical symmetry of the gyrostat is acted upon by a force inversely proportional to the cube of its distance from a fixed plane through the fixed point. The full result ($\lambda \neq 0$) can be used as an integrable model approximating real mechanical systems far from the singularity.

The special version ($\lambda = 0$), however, admits a full physical interpretation as the problem of the motion of a rigid-body gyrostat about a fixed point. The constant term Ck in $\boldsymbol{\mu}$ is the gyrostatic moment of a symmetric rotor kept along the axis of dynamical symmetry of the principal body. The variable part of the vector $\boldsymbol{\mu}$ can be viewed as a result of Lorentz' interaction between a distribution of electric charges fixed in the body and a uniform magnetic field in the direction of the vector $\boldsymbol{\gamma}$ (see [1]). The quadratic part of the potential can be interpreted as being due to a linearly non-uniform gravitational (or electric) field acting on the mass (or charge) distribution in the body. In interpreting the linear part of the potential we have the freedom to assume the presence of one or more interactions between the total mass of the body and a uniform gravity field, between the total charge and a uniform electric field or scalar interaction between the uniform magnetic field and a magnetized part of the body or, eqivalently steady currents in circuits carried by the body.

The same version $\lambda = 0$ can also be interpreted as a problem of motion by inertia of a rigid body bounded by a multi-connected surface in an infinite medium of ideal incompressible fluid in a state of vortex flow and at rest at infinity. In fact, the equations of motion of the latter problem were reduced in [9] to the form of equations of motion of a body about a fixed point. The system (1), (2) for $\lambda = 0$ is a special case of this form. In this interpretation the linear terms of the potential and the constant Ck (the gyrostatic moment) are related to the multiconnected character of the surface of the body and to circulations of the fluid through holes (perforations) in it [9, 12].

References

[1] Yehia H M 1986 *J. Mecan. Theor. Appl.* **5** 747

[2] Goriatchev D N 1900 *Mat. Sbornik Kr. Lub. Mat. Nauk* **21** (3) 431

[3] Chaplygin S A 1901 *Trudy Otd. Fiz. Nauk Obsh. Lub. Yest.* **10** (2) 32

[4] Whittaker E T 1944 *Treatise on Analytical Dynamics of Particles and Rigid Bodies* (New York: Dover)

[5] Leimanis E 1965 *The General Problem of Motion of Coupled Rigid Bodies About a Fixed Point* (Berlin: Springer)

[6] Goriatchev D N 1915 *Warsaw Univ. Izv.* (3) 1

[7] Sretensky L N 1963 *Vestn. Mosk. Univ. Mat.–Mekh.* (3) 60

[8] Yehia H M 1986 *J. Mecan. Theor. Appl.* **5** 55

[9] Yehia H M 1986 *J. Mecan. Theor. Appl.* **5** 755

[10] Goriatchev D N 1916 *Warsaw Univ. Izv.* (3) 1

[11] Yehia H M 1988 *Celest. Mech.* **41** 289

[12] Lamb H 1932 *Hydrodynamics* (Cambridge: Cambridge University Press)